

ON THE INDEPENDENT SUBSETS OF POWERS OF PATHS AND CYCLES

PIETRO CODARA, OTTAVIO M. D'ANTONA

ABSTRACT. In the first part of this work we provide a formula for the number of edges of the Hasse diagram of the independent subsets of the h^{th} power of a path ordered by inclusion. For $h = 1$ such a value is the number of edges of a Fibonacci cube. We show that, in general, the number of edges of the diagram is obtained by convolution of a Fibonacci-like sequence with itself.

In the second part we consider the case of cycles. We evaluate the number of edges of the Hasse diagram of the independent subsets of the h^{th} power of a cycle ordered by inclusion. For $h = 1$, and $n > 1$, such a value is the number of edges of a Lucas cube.

1. INTRODUCTION

For a graph \mathbf{G} we denote by $V(\mathbf{G})$ the set of its vertices, and by $E(\mathbf{G})$ the set of its edges.

Definition 1.1. For $n, h \geq 0$,

- (i) the h -power of a path, denoted by $\mathbf{P}_n^{(h)}$, is a graph with n vertices v_1, v_2, \dots, v_n such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{P}_n^{(h)})$ if and only if $|j - i| \leq h$;
- (ii) the h -power of a cycle, denoted by $\mathbf{Q}_n^{(h)}$, is a graph with n vertices v_1, v_2, \dots, v_n such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(\mathbf{Q}_n^{(h)})$ if and only if $|j - i| \leq h$ or $|j - i| \geq n - h$.

Thus, for instance, $\mathbf{P}_n^{(0)}$ and $\mathbf{Q}_n^{(0)}$ are the graphs made of n isolated nodes, $\mathbf{P}_n^{(1)}$ is the path with n vertices, and $\mathbf{Q}_n^{(1)}$ is the cycle with n vertices. Figures 1(a), and 1(b) show some powers of paths and cycles, respectively.

Definition 1.2. An *independent subset* of a graph \mathbf{G} is a subset of $V(\mathbf{G})$ not containing adjacent vertices.

Let $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ be the Hasse diagrams of the posets of independent subsets of $\mathbf{P}_n^{(h)}$, and $\mathbf{Q}_n^{(h)}$, respectively, ordered by inclusion. Clearly, $\mathbf{H}_n^{(0)} \cong \mathbf{M}_n^{(0)}$ is a Boolean lattice with n atoms (n -cube, for short).

Every independent subset S of $\mathbf{P}_n^{(h)}$ can be represented by a binary string $b_1 b_2 \dots b_n$, where, for $i = 1, \dots, n$, $b_i = 1$ if and only if $v_i \in S$. Specifically, each independent subset of $\mathbf{P}_n^{(h)}$ is associated with a binary string of length n such

2010 *Mathematics Subject Classification.* Primary: 68R05. Secondary: 11B39.

Key words and phrases. Independent subset, path, cycle, power of graph, Fibonacci cube, Lucas cube.

This research was partially supported by “Dote ricerca” – FSE, Regione Lombardia.

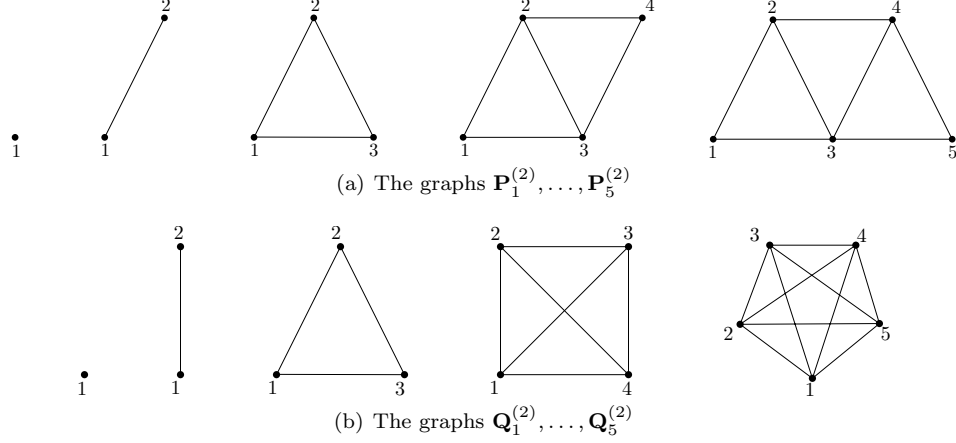


FIGURE 1. Some powers of paths and cycles.

that the distance between any two 1's of the string is greater than h . Following [MS02] (see also [Kla11]), a *Fibonacci string of order n* is a binary string of length n without (two) consecutive 1's. Recalling that the Hamming distance between two binary strings α and β is the number $H(\alpha, \beta)$ of bits where α and β differ, we can define the *Fibonacci cube of order n* , denoted Γ_n , as the graph (V, E) , where V is the set of all Fibonacci strings of order n and, for all $\alpha, \beta \in V$, $(\alpha, \beta) \in E$ if and only if $H(\alpha, \beta) = 1$. One can observe that for $h = 1$ the binary strings associated with independent subsets of $\mathbf{P}_n^{(h)}$ are *Fibonacci strings of order n* , and the Hasse diagram of the set of all such strings ordered bitwise is Γ_n . Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [Hsu93], and their combinatorial structure has been further investigated, *e.g.* in [KP07, MS02]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, *e.g.*, [IKR12a, Kla11]).

Remark. Consider the *generalized Fibonacci cubes* described in [IKR12a], *i.e.*, the graphs $B_n(\alpha)$ obtained from the n -cube B_n of all binary strings of length n by removing all vertices that contain the binary string α as a substring. In this notation the Fibonacci cube is $B_n(11)$. It is not difficult to see that $\mathbf{H}_n^{(h)}$ cannot be expressed, in general, in terms of $B_n(\alpha)$. Instead we have:

$$\mathbf{H}_n^{(2)} = B_n(11) \cap B_n(101), \quad \mathbf{H}_n^{(3)} = B_n(11) \cap B_n(101) \cap B_n(1001), \quad \dots,$$

where $B_n(\alpha) \cap B_n(\beta)$ is the subgraph of B_n obtaining by removing all strings that contain either α or β .

A similar argument can be carried out in the case of cycles. Indeed, every independent subset S of $\mathbf{Q}_n^{(h)}$ can be represented by a circular binary string (*i.e.*, a sequence of 0's and 1's with the first and last bits considered to be adjacent) $b_1 b_2 \dots b_n$, where, for $i = 1, \dots, n$, $b_i = 1$ if and only if $v_i \in S$. Thus, each independent subset of $\mathbf{Q}_n^{(h)}$ is associated with a circular binary string of length n such that the distance between any two 1's of the string is greater than h . A *Lucas cube of order n* , denoted Λ_n , is defined as the graph whose vertices are the binary strings of length n without either two consecutive 1's or a 1 in the first and in the

last position, and in which the vertices are adjacent when their Hamming distance is exactly 1 (see [MPCZS01]). For $h = 1$ the Hasse diagram of the set of all circular binary strings associated with independent subsets of $\mathbf{Q}_n^{(h)}$ ordered bitwise is Λ_n . A generalization of the notion of Lucas cubes has been proposed in [IKR12b].

Remark. Consider the *generalized Lucas cubes* described in [IKR12b], that is, the graphs $B_n(\widehat{\alpha})$ obtained from the n -cube B_n of all binary strings of length n by removing all vertices that have a *circular containing* α as a substring (*i.e.*, such that α is contained in the circular binary strings obtained by connecting first and last bits of the string). In this notation the Lucas cube is $B_n(\widehat{11})$. It is not difficult to see that $\mathbf{M}_n^{(h)}$ cannot be expressed, in general, in terms of $B_n(\widehat{\alpha})$. Instead we have:

$$\mathbf{M}_n^{(2)} = B_n(\widehat{11}) \cap B_n(\widehat{101}), \mathbf{M}_n^{(3)} = B_n(\widehat{11}) \cap B_n(\widehat{101}) \cap B_n(\widehat{1001}), \dots$$

As far as we now, our $\mathbf{H}_n^{(h)}$, and $\mathbf{M}_n^{(h)}$ are new generalizations of Fibonacci and Lucas cubes, respectively.

In the first part of the paper we evaluate $p_n^{(h)}$, *i.e.*, the number of *independent subsets* of $\mathbf{P}_n^{(h)}$, and $H_n^{(h)}$, *i.e.*, the number of edges of $\mathbf{H}_n^{(h)}$. Our main result (Theorem 3.4) is that, for $n, h \geq 0$, the sequence $H_n^{(h)}$ is obtained by convolving the sequence $\underbrace{1, \dots, 1}_h, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \dots$ with itself.

In the second part of the paper we derive similar results for $q_n^{(h)}$, *i.e.*, the number of *independent subsets* of $\mathbf{Q}_n^{(h)}$, and $M_n^{(h)}$, *i.e.*, the number of edges of $\mathbf{M}_n^{(h)}$.

2. THE INDEPENDENT SUBSETS OF POWERS OF PATHS

For $n, h, k \geq 0$, we denote by $p_{n,k}^{(h)}$ the number of independent k -subsets of $\mathbf{P}_n^{(h)}$.

Remark. For $h = 1$, $p_{n,k}^{(h)}$ counts the number of binary strings $\alpha \in \Gamma_n$ such that $H(\alpha, 00 \dots 0) = k$.

Lemma 2.1. *For $n, h, k \geq 0$,*

$$p_{n,k}^{(h)} = \binom{n - hk + h}{k}.$$

This result is Theorem 1 of [Hog70]. Below we write down a different proof.

Proof. By Definition 1.2, any two elements v_i, v_j of an independent subset of $\mathbf{P}_n^{(h)}$ must satisfy $|j - i| > h$. It is straightforward to check that whenever $n - hk - h < 0$, $p_{n,k}^{(h)} = 0 = \binom{n - hk + h}{k}$. It is also immediate to see that when $n = h = 0$ our lemma holds true.

Suppose now $n - hk - h \geq 0$. We can complete the proof of our lemma by establishing a bijection between independent k -subset of $\mathbf{P}_n^{(h)}$ and k -subsets of a set with $(n - hk + h)$ elements. Let \mathcal{K} be the set of all k -subsets of a set $B = \{b_1, b_2, \dots, b_{n - hk + h}\}$, and \mathcal{I}_k the set of all independent k -subsets of $\mathbf{P}_n^{(h)}$. Consider the map $f : \mathcal{K} \rightarrow \mathcal{I}_k$ such that, for any $S = \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\} \in \mathcal{K}$, with $1 \leq i_1 < i_2 < \dots < i_k \leq n - hk + h$,

$$f(\{b_{i_1}, b_{i_2}, \dots, b_{i_j}, \dots, b_{i_k}\}) = \{v_{i_1}, v_{i_2+h}, \dots, v_{i_j+(j-1)h}, \dots, v_{i_k+(k-1)h}\}.$$

Claim 1. The map f associates with each k -subset $S = \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\} \in \mathcal{K}$ an independent k -subset of $\mathbf{P}_n^{(h)}$.

To see this we first remark that $f(S)$ is a k -subset of $V(\mathbf{P}_n^{(h)})$. Furthermore, for each pair $b_{i_j}, b_{i_{j+t}} \in S$, with $t > 0$, we have

$$i_{j+t} + (j+t-1)h - (i_j + (j-1)h) = i_{j+t} - i_j + th > h.$$

Hence, by Definition 1.1, $(f(b_{i_j}), f(b_{i_{j+t}})) = (v_{i_j+(j-1)h}, v_{i_{j+t}+(j+t-1)h}) \notin E(\mathbf{P}_n^{(h)})$. Thus, $f(S)$ is an independent subset of $\mathbf{P}_n^{(h)}$.

Claim 2. The map f is bijective.

It is easy to see that f is injective. Then, we consider the map $f^{-1} : \mathcal{I}_k \rightarrow \mathcal{K}$ such that, for any $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \in \mathcal{I}$, with $1 \leq i_1 < i_2 < \dots < i_k \leq n$,

$$f^{-1}(\{v_{i_1}, v_{i_2}, \dots, v_{i_j}, \dots, v_{i_k}\}) = \{b_{i_1}, b_{i_2-h}, \dots, b_{i_j-(j-1)h}, \dots, b_{i_k-(k-1)h}\}.$$

Following the same steps as for f , one checks that f^{-1} is injective. Thus, f is surjective.

By Claims 1 and 2 we have established a bijection between independent k -subsets of $\mathbf{P}_n^{(h)}$ and k -subsets of a set with $(n - hk + h) \geq 0$ elements. The lemma is proved. \square

The coefficients $p_{n,k}^{(h)}$ also enjoy the following property: $p_{n,k}^{(h)} = p_{n-k+1,k}^{(h-1)}$.

For $n, h \geq 0$, the number of all independent subsets of $\mathbf{P}_n^{(h)}$ is

$$p_n^{(h)} = \sum_{k \geq 0} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n - hk + h}{k}.$$

Remark. Denote by F_n the n^{th} element of the Fibonacci sequence $F_1 = 1, F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$ is the number of elements of the Fibonacci cube of order n .

The following, simple fact is crucial for our work.

Lemma 2.2. For $n, h \geq 0$,

$$p_n^{(h)} = \begin{cases} n+1 & \text{if } n \leq h+1, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h+1. \end{cases}$$

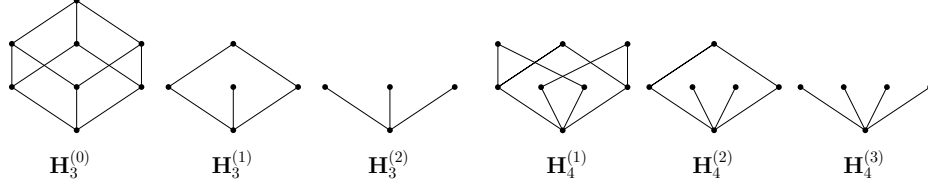
A proof of this Lemma can be also obtained using the first part of [Hog70, Proof of Theorem 1].

Proof. For $n \leq h+1$, by Definition 1.2, the independent subsets of $\mathbf{P}_n^{(h)}$ have no more than 1 element. Thus, there are $n+1$ independent subsets of $\mathbf{P}_n^{(h)}$.

Consider the case $n > h+1$. Let \mathcal{I} be the set of all independent subsets of $\mathbf{P}_n^{(h)}$, let \mathcal{I}_{in} be the set of the independent subsets of $\mathbf{P}_n^{(h)}$ that contain v_n , and let $\mathcal{I}_{out} = \mathcal{I} \setminus \mathcal{I}_{in}$. The elements of \mathcal{I}_{out} are in one-to-one correspondence with the $p_{n-1}^{(h)}$ independent subsets of $\mathbf{P}_{n-1}^{(h)}$, and those of \mathcal{I}_{in} are in one-to-one correspondence with the $p_{n-h-1}^{(h)}$ independent subsets of $\mathbf{P}_{n-h-1}^{(h)}$. \square

3. THE POSET OF INDEPENDENT SUBSETS OF POWERS OF PATHS

Figure 2 shows a few Hasse diagrams $\mathbf{H}_n^{(h)}$. Notice that, as stated in the introduction, for each n , $\mathbf{H}_n^{(1)}$ is the Fibonacci cube Γ_n .

FIGURE 2. Some $\mathbf{H}_n^{(h)}$.

Let $H_n^{(h)}$ be the number of edges of $\mathbf{H}_n^{(h)}$. Noting that in $\mathbf{H}_n^{(h)}$ each non-empty independent k -subset covers exactly k independent $(k-1)$ -subsets, we can write

$$(1) \quad H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k p_{n,k}^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \binom{n-hk+h}{k}.$$

Remark. For $h=1$, $H_n^{(h)}$ counts the number of edges of Γ_n .

Let now $T_{k,i}^{(n,h)}$ be the number of independent k -subsets of $\mathbf{P}_n^{(h)}$ containing the vertex v_i , and let, for $h, k \geq 0$, $n \in \mathbb{Z}$, $\bar{p}_{n,k}^{(h)} = \begin{cases} p_{0,k}^{(h)} & \text{if } n < 0, \\ p_{n,k}^{(h)} & \text{if } n \geq 0. \end{cases}$

Lemma 3.1. For $n, h, k \geq 0$, and $1 \leq i \leq n$,

$$T_{k,i}^{(n,h)} = \sum_{r=0}^{k-1} \bar{p}_{i-h-1,r}^{(h)} \bar{p}_{n-i-h,k-1-r}^{(h)}.$$

Proof. No independent subset of $\mathbf{P}_n^{(h)}$ containing v_i contains any of the elements $v_{i-h}, \dots, v_{i-1}, v_{i+1}, \dots, v_{i+h}$. Let r and s be non-negative integers whose sum is $k-1$. Each independent k -subset of $\mathbf{P}_n^{(h)}$ containing v_i can be obtained by adding v_i to a $(k-1)$ -subset $R \cup S$ such that

- (a) $R \subseteq \{v_1, \dots, v_{i-h-1}\}$ is an independent r -subset of $\mathbf{P}_n^{(h)}$;
- (b) $S \subseteq \{v_{i+h+1}, \dots, v_n\}$ is an independent s -subset of $\mathbf{P}_n^{(h)}$.

Viceversa, one can obtain each of this pairs of subsets by removing v_i from an independent k -subset of $\mathbf{P}_n^{(h)}$ containing v_i . Thus, $T_{k,i}^{(n,h)}$ is obtained by counting independently the subsets of type (a) and (b). Noting that the subsets of type (b) are in bijection with the independent s -subsets of $\mathbf{P}_{n-i-h}^{(h)}$, the lemma is proved. \square

Remark. $T_{k,i}^{(n,1)}$ counts the number of strings $\alpha = b_1 b_2 \dots b_n \in \Gamma_n$ such that: (i) $H(\alpha, 00 \dots 0) = k$, and (ii) $b_i = 1$.

In order to obtain our main result, we prepare a lemma.

Lemma 3.2. *For positive n ,*

$$\sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^n T_{k,i}^{(n,h)} = H_n^{(h)}.$$

Proof. The inner sum counts the number of k -subsets exactly k times, one for each element of the subset. That is, $\sum_{i=1}^n T_{k,i}^{(n,h)} = k p_{n,k}^{(h)}$. The lemma follows directly from Equation (1). \square

Next we introduce a family of Fibonacci-like sequences.

Definition 3.3. For $h \geq 0$, and $n \geq 1$, we define the h -Fibonacci sequence $\mathcal{F}^{(h)} = \{F_n^{(h)}\}_{n \geq 1}$ whose elements are

$$F_n^{(h)} = \begin{cases} 1 & \text{if } n \leq h+1, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h+1. \end{cases}$$

From Lemma 2.2, and setting for $h \geq 0$, and $n \in \mathbb{Z}$, $\bar{p}_n^{(h)} = \begin{cases} p_0^{(h)} & \text{if } n < 0, \\ p_n^{(h)} & \text{if } n \geq 0, \end{cases}$ we have that,

$$(2) \quad F_i^{(h)} = \bar{p}_{i-h-1}^{(h)}, \quad \text{for each } i \geq 1.$$

Thus, our Fibonacci-like sequences are obtained by adding a prefix of h ones to the sequence $p_0^{(h)}, p_1^{(h)}, \dots$. Therefore, we have:

- $\mathcal{F}^{(0)} = 1, 2, 4, \dots, 2^n, \dots$;
- $\mathcal{F}^{(1)}$ is the Fibonacci sequence;
- more generally, $\mathcal{F}^{(h)} = \underbrace{1, \dots, 1}_h, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \dots$.

In the following, we use the discrete convolution operation $*$, as follows.

$$(3) \quad \left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)} \right) (n) \doteq \sum_{i=1}^n F_i^{(h)} F_{n-i+1}^{(h)}$$

Theorem 3.4. *For $n, h \geq 0$, the following holds.*

$$H_n^{(h)} = \left(\mathcal{F}^{(h)} * \mathcal{F}^{(h)} \right) (n).$$

Proof. The sum $\sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)}$ counts the number of independent subsets of $\mathbf{P}_n^{(k)}$ containing v_i . We can also obtain such a value by counting the independent subsets of both $\{v_1, \dots, v_{i-h-1}\}$, and $\{v_{i+h+1}, \dots, v_n\}$. Thus, we have:

$$\sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.$$

Using Lemma 3.2 we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^n T_{k,i}^{(n,h)} = \sum_{i=1}^n \sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \sum_{i=1}^n \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.$$

By Equation (2) we have $\sum_{i=1}^n \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} = \sum_{i=1}^n F_i^{(h)} F_{n-i+1}^{(h)}$. By (3), the theorem is proved. \square

Remark. For $h = 1$, we obtain the number of edges of Γ_n by using Fibonacci numbers:

$$H_n^{(h)} = \sum_{i=1}^n F_i F_{n-i+1}.$$

The latter result is [Kla05, Proposition 3].

4. THE INDEPENDENT SUBSETS OF POWERS OF CYCLES

For $n, h, k \geq 0$, we denote by $q_{n,k}^{(h)}$ the number of independent k -subsets of $\mathbf{Q}_n^{(h)}$.

Remark. For $h = 1$, $n > 1$, $q_{n,k}^{(h)}$ counts the number of binary strings $\alpha \in \Lambda_n$ such that $H(\alpha, 00 \cdots 0) = k$.

Lemma 4.1. *For $n, h \geq 0$, and $k > 1$,*

$$q_{n,k}^{(h)} = \frac{n}{k} \binom{n - hk - 1}{k - 1}.$$

Moreover, $q_{n,0}^{(h)} = 1$, and $q_{n,1}^{(h)} = n$, for each $n, h \geq 0$.

Proof. Fix an element $v_i \in V(\mathbf{Q}_n^{(h)})$, and let $n > 2h$. Any independent subset of $\mathbf{Q}_n^{(h)}$ containing v_i does not contain the h elements preceding v_i and the h elements following v_i . Thus, the number of independent k -subsets of $\mathbf{Q}_n^{(h)}$ containing v_i equals

$$p_{n-2h-1,k-1}^{(h)} = \binom{n - hk - 1}{k - 1}.$$

The total number of independent k -subsets of $\mathbf{Q}_n^{(h)}$ is obtained by multiplying $p_{n-2h-1,k-1}^{(h)}$ by n , then dividing it by k (each subset is counted k times by the previous proceeding). The case $n \leq 2h$, as well as the cases $k = 0, 1$, can be easily verified. \square

For $n, h \geq 0$, the number of all independent subsets of $\mathbf{Q}_n^{(h)}$ is

$$(4) \quad q_n^{(h)} = \sum_{k \geq 0} q_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} q_{n,k}^{(h)},$$

Remark. Denote by L_n the n^{th} element of the Lucas sequence $L_1 = 1$, $L_2 = 3$, and $L_i = L_{i-1} + L_{i-2}$, for $i > 2$. Then, for $n > 1$, $q_n^{(1)} = L_n$ is the number of elements of the Lucas cube of order n .

The coefficients $q_n^{(h)}$ satisfy a recursion that closely resemble that of Lemma 2.2.

Lemma 4.2. *For $n, h \geq 0$,*

$$(5) \quad q_n^{(h)} = \begin{cases} n + 1 & \text{if } n \leq 2h + 1, \\ q_{n-1}^{(h)} + q_{n-h-1}^{(h)} & \text{if } n > 2h + 1. \end{cases}$$

Proof. The case $n \leq 2h + 1$ can be easily checked. Let $n > 3h + 2$, and let \mathcal{I} be the set of the independent subsets of $\mathbf{Q}_n^{(h)}$. Let \mathcal{I}_{in} be the subset of these subsets that (i) do not contain v_n , and that (ii) contain no one of the following pairs: $(v_1, v_{n-h}), (v_2, v_{n-h+1}), \dots, (v_h, v_{n-1})$. Furthermore let \mathcal{I}_{out} be the subset of the remaining independent subsets of $\mathbf{Q}_n^{(h)}$.

It is easy to see that the elements of \mathcal{I}_{in} are exactly the independent subsets of $\mathbf{Q}_{n-1}^{(h)}$. Indeed, v_n is not a vertex of $\mathbf{Q}_{n-1}^{(h)}$ and the vertices of pairs (v_1, v_{n-h}) , (v_2, v_{n-h+1}) , \dots , (v_h, v_{n-1}) are connected in $\mathbf{Q}_{n-1}^{(h)}$. On the other hand, to show that

$$|\mathcal{I}_{out}| = q_{n-h-1}^{(h)}$$

we argue as follows. First we recall (see the proof of Lemma 4.1) that the number of independent k -subsets of $\mathbf{Q}_n^{(h)}$ that contain v_n is $p_{n-2h-1, k-1}^{(h)}$. Secondly we obtain that the number of independent k -subsets of $\mathbf{Q}_n^{(h)}$ containing one of the pairs (v_1, v_{n-h}) , (v_2, v_{n-h+1}) , \dots , (v_h, v_{n-1}) is $h p_{n-3h-2, k-2}^{(h)}$. To see this, consider the pair (v_1, v_{n-h}) . The independent subsets containing such a pair do not contain the h vertices from v_{n-h+1} to v_n , do not contain the h vertices from v_2 to v_{h+1} , and do not contain the h vertices from v_{n-2h} to v_{n-h-1} . Thus, the removal of such vertices and of the vertices v_1 and v_{n-h} turns $\mathbf{Q}_n^{(h)}$ into $\mathbf{P}_{n-3h-2}^{(h)}$. Hence we can obtain all the independent k -subsets of $\mathbf{Q}_n^{(h)}$ that contain the pair (v_1, v_{n-h}) by simply adding these two vertices to one of the $p_{n-3h-2, k-2}^{(h)}$ independent $k-2$ -subsets of $\mathbf{P}_{n-3h-2}^{(h)}$. Same reasoning can be carried out for any other one of the pairs: (v_2, v_{n-h+1}) , \dots , (v_h, v_{n-1}) .

Using Lemmas 2.1 and 4.1 one can easily derive that

$$p_{n-2h-1, k-1}^{(h)} + h p_{n-3h-2, k-2}^{(h)} = q_{n-h-1, k-1}^{(h)}.$$

Hence, we derive the size of \mathcal{I}_{out} :

$$|\mathcal{I}_{out}| = q_{n-h-1}^{(h)} = \sum_{k \geq 1} p_{n-2h-1, k-1}^{(h)} + h \sum_{k \geq 2} p_{n-3h-2, k-2}^{(h)}.$$

Summing up we have shown that $|\mathcal{I}| = |\mathcal{I}_{in}| + |\mathcal{I}_{out}|$, that is

$$q_n^{(h)} = q_{n-1}^{(h)} + q_{n-h-1}^{(h)}.$$

The proof of the case $2h+1 < n \leq 3h+2$ is obtained in a similar way, observing that $|\mathcal{I}_{out}| = n-h$, and that $n-h-1 \leq 2h+1$. \square

5. THE POSET OF INDEPENDENT SUBSETS OF POWERS OF CYCLES

Figure 3 shows a few Hasse diagrams $\mathbf{M}_n^{(h)}$. Notice that, as stated in the introduction, for each n , $\mathbf{M}_n^{(1)}$ is the Lucas cube Λ_n .

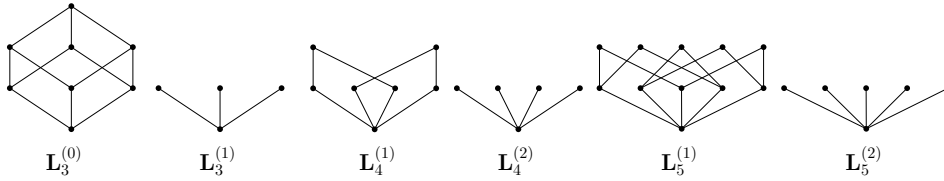


FIGURE 3. Some $\mathbf{M}_n^{(h)}$.

Let $M_n^{(h)}$ be the number of edges of $\mathbf{M}_n^{(h)}$. As done in Section 3 for the case of paths, we immediately provide a formula for $M_n^{(h)}$:

$$(6) \quad M_n^{(h)} = \sum_{k=0}^{\lceil n/h+1 \rceil} k q_{n,k}^{(h)} = n \sum_{k=0}^{\lceil n/h+1 \rceil} \binom{n-hk-1}{k-1}.$$

Remark. For $h = 1$, $n > 1$, $M_n^{(h)}$ counts the number of edges of Λ_n . As shown in [MPCZS01, Proposition 4(ii)], $M_n^{(h)} = nF_{n-1}$.

As shown in the proof of Lemma 4.1, the value

$$p_{n-2h-1,k-1}^{(h)} = \binom{n-hk-1}{k-1}$$

is the analogue of the coefficient $T_{k,i}^{(n,h)}$: in the case of cycles we have no dependencies on i , because each choice of vertex is equivalent. We can obtain $M_n^{(h)}$ in terms of a fibonacci-like sequence, as follows.

Proposition 5.1. *For $n > h \geq 0$, the following holds.*

$$M_n^{(h)} = nF_{n-h}^{(h)}.$$

Proof. Using Equation (2) we obtain:

$$M_n^{(h)} = n \sum_{k=1}^{\lceil n/(h+1) \rceil} \bar{p}_{n-2h-1,k-1}^{(h)} = n \bar{p}_{n-2h-1}^{(h)} = nF_{n-h}^{(h)}.$$

□

REFERENCES

- [Hog70] Verner E. Hoggatt, Jr., *Combinatorial problems for generalized Fibonacci numbers*, Fibonacci Quart. **8** (1970), no. 5, 456–462.
- [Hsu93] Wen-Jing Hsu, *Fibonacci cubes-a new interconnection technology*, IEEE Trans. Parallel Distrib. Syst. **4** (1993), no. 1, 3–12.
- [IKR12a] Aleksandar Ilić, Sandi Klavžar, and Yoomi Rho, *Generalized Fibonacci cubes*, Discrete Math. **312** (2012), no. 1, 2–11.
- [IKR12b] ———, *Generalized Lucas cubes*, Appl. Anal. Discrete Math. **6** (2012), no. 1, 82–94.
- [Kla05] Sandi Klavžar, *On median nature and enumerative properties of Fibonacci-like cubes*, Discrete Math. **299** (2005), no. 1-3, 145–153.
- [Kla11] ———, *Structure of fibonacci cubes: a survey*, Journal of Combinatorial Optimization (2011), 1–18 (English).
- [KP07] Sandi Klavžar and Iztok Peterin, *Edge-counting vectors, Fibonacci cubes, and Fibonacci triangle*, Publ. Math. Debrecen **71** (2007), no. 3-4, 267–278.
- [MPCZS01] Emanuele Munarini, Claudio Perelli Cippo, and Norma Zagaglia Salvi, *On the Lucas cubes*, Fibonacci Quart. **39** (2001), no. 1, 12–21.
- [MS02] Emanuele Munarini and Norma Zagaglia Salvi, *Structural and enumerative properties of the Fibonacci cubes*, Discrete Math. **255** (2002), no. 1-3, 317–324.

(P. Codara, O. M. D’Antona) DIPARTIMENTO DI INFORMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA COMELICO 39/41, I-20135, MILAN, ITALY

E-mail address: {codara,dantona}@dico.unimi.it